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LETTER TO THE EDITOR

Tricritical behaviour in an Ising system and the Potts model

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Abstract. The susceptibility of the Ising $S = 1$ model with biquadratic interactions is expanded in a high temperature series on the FCC lattice for a range of the biquadratic interaction parameter that includes the Potts model. A tricritical point and a point above which the magnetization M remains zero are approximately located. The Potts model seems to fall in the region of the first-order phase transition in M.

The nature of the phase transition of the Potts model has been under discussion recently. Mean field theory predicts a first-order phase transition as does the work of Golner (1973) in three dimensions using the Wilson approximate recursion formula. The latter calculation imposes $\eta = 0$ which could be inapplicable to this model. Straley and Fisher (1973) find a second-order phase transition for the two-dimensional case.

The Potts model is a three-component model (Potts 1952) where the energy of interacting nearest neighbours on the lattice is ϵ_0 when they are of the same species, and ϵ_1 when they belong to different species.

An Ising spin 1 model with biquadratic interactions has been investigated by mean field theory by Rys (1969), Oran (1972), Chen and Levy (1973) and Blume and Hsieh (1969). This model has the following hamiltonian:

$$\mathscr{H} = -J\sum_{\langle ij\rangle} S_{zi}S_{zj} - J^Q\sum_{\langle ij\rangle}(S_{zi}^2-\tfrac{2}{3})(S_{zj}^2-\tfrac{2}{3}) - h\sum_i S_{zi} - \zeta\sum_i(S_{zi}^2-\tfrac{2}{3}). \tag{1}$$

The Potts model can be represented by the following hamiltonian:

$$\mathscr{H} = (\tfrac{1}{3}\epsilon_0+\tfrac{2}{3}\epsilon_1)+[\tfrac{1}{2}(\epsilon_0-\epsilon_1)]\left(\sum_{\langle ij\rangle} S_{zi}S_{zj}+3\sum_{\langle ij\rangle}(S_{zi}^2-\tfrac{2}{3})(S_{zj}^2-\tfrac{2}{3})\right) \tag{2}$$

where the three species are $S_z = 1, 0, -1$ so (1) in zero fields $h = \zeta = 0$ with $j^Q = 3J$ is the Potts hamiltonian.

The order parameters are $M = \Sigma_i S_{zi}$ and $Q = \Sigma_i(S_{zi}^2-\tfrac{2}{3})$. Oran (1972) and Chen and Levy (1973) find that MFT predicts a second-order phase transition for $0 \leqslant J^Q/J < \tfrac{3}{2}$, a first-order phase transition for $\tfrac{3}{2} < J^Q/J < 3$ for both M and Q making $J^Q = \tfrac{3}{2}J$ a tricritical point, and for $J^Q/J > 3$ $M \equiv 0$ and Q has a first-order phase transition. Clearly these points of changes will vary with dimensionality. If we may assume that MFT gives the results for $d = 4$, Fisher and Straley's result puts the tricritical point in $d = 2$ at $J^Q/J > 3$. In $d = 3$ we may expect the tricritical point to be between the values for $d = 2$ and $d = 4$ and its location will determine if the

Potts model with $J^Q = 3J$ has a second- or first-order transition. This we attempt to do here.

High temperature series expansions were calculated for the initial magnetic susceptibility with the hamiltonian (3) using methods explained by Oitmaa (1971):

$$\mathscr{H} = J \sum_{\langle ij \rangle} S_{zi}S_{zj} - J^Q \sum_{\langle ij \rangle} S_{zi}^2 S_{zj}^2 - \mu \sum_{i} S_{zi}^2. \tag{3}$$

The susceptibility is expressed as

$$kT\chi = \tau + \sum_{\substack{s,d=0 \\ s+d\neq 0}}^{\infty} M_{s,d}(\tau) X^s Y^d$$

where $M_{s,d}(\tau)$ are polynomials in τ (see appendix) and

$$X = \exp(J^Q/kT)\sinh(J/kT)$$
$$Y = \exp(J^Q/kT)\cosh(J/kT) - 1$$
$$\tau = 2\exp(\mu/kT)/[2\exp(\mu/kT) + 1].$$

For the model in equation (1) $\mu = -\frac{2}{3}qJ^Q$ where q is the coordination number. For a series of fixed values of J^Q/J we obtained χ as a series in $K = J/kT$ and analysed it by standard techniques. As the ratio J^Q/J increases we see the critical temperature as estimated from the series first rises gradually then drops gradually and then drops very

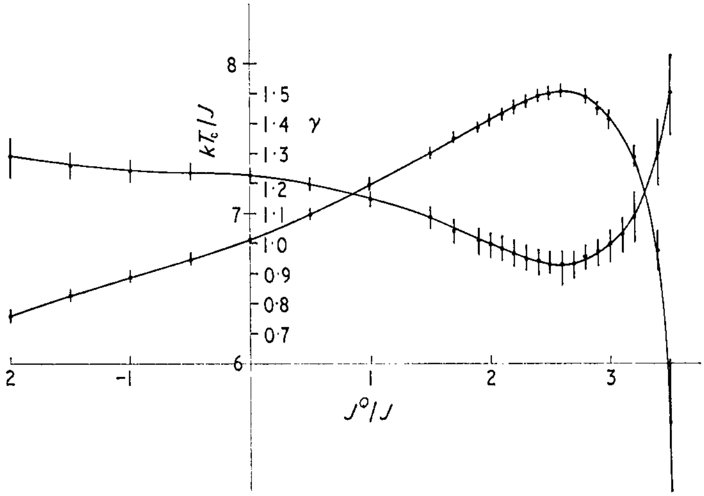

Figure 1. Plot of γ and kT_c/J against J^Q/J. The curve with the minimum is γ.

quickly to zero. The last rapid drop we identify with the magnetization remaining identically zero at all temperatures and only Q ordering. The point at which the critical temperature turns from increasing to decreasing with increased biquadratic interaction indicating an onset of instability we identify as the tricritical point with some hesitation.

There are two arguments for this identification. Firstly, the tricritical point in MFT is also the turning point for T_c from constant to decreasing. Secondly, the value we estimate for γ in a region near that point is very close to 1, and $\gamma = 1$ is what one would expect to find near a tricritical point. But this applies to $1{\cdot}5 < J^Q/J < 3{\cdot}2$.

Our estimates are that the tricritical point is at $J^Q/J = 2{\cdot}63 \pm 0{\cdot}05$ and for J^Q/J greater than $3{\cdot}8 \pm 0{\cdot}2$ the magnetization M remains zero at all temperatures. Constant γ over $-2 < J^Q/J < 1$ is consistent with our results.

Work on the other susceptibility, namely $\chi' = \partial^2 \ln Z/\partial^2\zeta$ is in progress. Calculations of the low- and high-temperature series for the free energy which will enable us to locate the tricritical point more accurately and to obtain the full phase diagrams are also under way.

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Appendix

$$M_{1,0} = 12\tau^2$$
$$M_{0,1} = 12\tau^2 - 12\tau^3$$
$$M_{2,0} = 132\tau^3$$
$$M_{1,1} = 264\tau^3 - 276\tau^4$$
$$M_{0,2} = 198\tau^3 - 474\tau^4 + 276\tau^5$$
$$M_{3,0} = 24\tau^3 + 1380\tau^4$$
$$M_{2,1} = 48\tau^3 + 4128\tau^4 - 4440\tau^5$$
$$M_{1,2} = 48\tau^3 + 5532\tau^4 - 13320\tau^5 + 7752\tau^6$$
$$M_{0,3} = 24\tau^3 + 3664\tau^4 - 14788\tau^5 + 18852\tau^6 - 7752\tau^7$$
$$M_{4,0} = 612\tau^4 + 14040\tau^5$$
$$M_{3,1} = 2184\tau^4 + 54240\tau^5 - 60636\tau^6$$
$$M_{2,2} = 2664\tau^4 + 102696\tau^5 - 252192\tau^6 + 147228\tau^7$$
$$M_{1,3} = 2184\tau^4 + 114888\tau^5 - 465744\tau^6 + 588912\tau^7 - 240252\tau^8$$
$$M_{0,4} = 1092\tau^4 + 70713\tau^5 - 421113\tau^6 + 864606\tau^7 - 755550\tau^8 + 240252\tau^9$$
$$M_{5,0} = 72\tau^4 + 104888\tau^5 + 140556\tau^6$$
$$M_{4,1} = 432\tau^4 + 51024\tau^5 + 648768\tau^6 - 758832\tau^7$$
$$M_{3,2} = 720\tau^4 + 97776\tau^5 + 1499292\tau^6 - 3898440\tau^7 + 2309076\tau^8$$
$$M_{2,3} = 576\tau^4 + 96480\tau^5 + 2325696\tau^6 - 9737928\tau^7 + 12326976\tau^8 - 5012328\tau^9$$
$$M_{1,4} = 360\tau^4 + 73200\tau^5 + 2349120\tau^6 - 14233800\tau^7 + 28977420\tau^8 - 25061640\tau^9 + 7895352\tau^{10}$$
$$M_{0,5} = 144\tau^4 + 36456\tau^5 + 1372872\tau^6 - 11373132\tau^7 + 33145596\tau^8 - 45737412\tau^9 + 30450828\tau^{10} - 7895352\tau^{11}$$
$$M_{6,0} = 3156\tau^5 + 149712\tau^6 + 1393464\tau^7$$
$$M_{5,1} = 24\tau^4 + 18624\tau^5 + 904920\tau^6 + 7322640\tau^7 - 9001788\tau^8$$
$$M_{4,2} = 72^4 + 45636\tau^5 + 2315556\tau^6 + 19109868\tau^7 - 53910756\tau^8 + 32586144\tau^9$$

$$M_{3,3} = 80\tau^4+56056\tau^5+3335856\tau^6+34725784\tau^7-161531608\tau^8+208923840\tau^9 -85524048\tau^{10}$$

$$M_{2,4} = 48\tau^4+39816\tau^5+2951088\tau^6+49217424\tau^7-316791492\tau^8+647986056\tau^9 -558397224\tau^{10}+174994944\tau^{11}$$

$$M_{1,5} = 24\tau^4+23616\tau^5+2155320\tau^6+46645512\tau^7-402298836\tau^8+1166561352\tau^9 -1593317496\tau^{10}+1049969664^{11}-269739168\tau^{12}$$

$$M_{0,6} = 8\tau^4+9832\tau^5+1067820\tau^6+26132222\tau^7-295409106\tau^8+1143120238\tau^9 -2202685594\tau^{10}+2290236772\tau^{11}-1232211360\tau^{12}+269739168\tau^{13}$$

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